Exercise 60

Determine whether $f'(0)$ exists.

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Solution

Recall that the derivative of f at $x = a$ is defined by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Plug in $a = 0$.

$$\begin{aligned} f'(0) &= \lim_{h \to 0} \frac{f(h) - f(0)}{h} \\ &= \lim_{h \to 0} \frac{h^2 \sin \frac{1}{h} - 0}{h} \\ &= \lim_{h \to 0} \frac{h^2 \sin \frac{1}{h}}{h} \\ &= \lim_{h \to 0} h \sin \frac{1}{h} \\ &= \lim_{h \to 0} \frac{\sin \frac{1}{h}}{\frac{1}{h}} \end{aligned}$$

Make the substitution,

$$u = \frac{1}{h},$$

in the limit. As h goes to zero, u becomes infinite.

$$f'(0) = \lim_{u \to \pm\infty} \frac{\sin u}{u} = 0$$

The limit exists, so $f'(0)$ exists.